Higher order iterative methods for solving nonlinear equations f(x)=0

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Recently, due to the development of various computer software and hardware many iterative methods have been developed to approximate a solution to nonlinear equations

$$f(x)=0. \qquad (1)$$

We present some new modifications in the last nine years of Newton-Raphson method, also we derived some new methods with higher order convergence for solving nonlinear equations (1) which are presented in this lecture.

Iterative methods are based on the idea of successive approximations, i.e., starting with one or more initial approximations to the root, we obtain a sequence of approximations or iterates $\{x_k\}$, which in the limit converges to the root. The methods give only one root at a time.

Definition 1:

A sequence of iterates $\{x_k\}$ is said to converge to the root α, if $\lim_{k\to\infty}|x_k-\alpha|=0$ or $\lim_{k\to\infty} x_k=\alpha$.

In practice, except in rare cases, it's not possible to find α which satisfies the given equation exactly.

Definition 2: (Order of convergence)

Assume that a sequence of iterates $\{x_k\}_{k=0}^{\infty}$ converges to α and $e_n = x_k - \alpha$ for $k \geq 0$. If two positive constants $M \neq 0$ and $q > 0$ exist, and $\lim_{k\to\infty} \frac{|\alpha - x_{k+1}|}{|\alpha - x_k|^q} = \lim_{k\to\infty} \frac{|e_{k+1}|}{|e_n|^q} = M$.

Then the sequence is said to converge to α with order of convergence q. The number M is called **asymptotic error constant.**

In recent years much attention has been given to develop several iterative type methods for solving non-linear equations. Hereunder we present the methods, but we focus on our new methods

The most popular and widely method for finding zeros of non-linear equations (1) is Newton's method for simple root which is defined by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)},$$

and Newton's method for multiple roots is defined by

$$x_{n+1} = x_n - m\frac{f(x_n)}{f'(x_n)}$$, (m is the multiplicity of the root),

These are important and basic methods, which converge quadratically.

In **2000**, ***Weerakoon*** and ***Fernando***, suggest an improvement of Newton method as follows:

From Newton's theorem,

$$f(x) = f(x_n) + \int_{x_n}^{x} f'(\lambda)d\lambda . \qquad (2)$$

In (2), the indefinite integral approximated by trapezoid rule, i.e.,

$$\int_{x_n}^{x} f'(\lambda)d\lambda \approx \frac{1}{2}(x - x_n)[f'(x) + f'(x)]. \qquad (3)$$

Thus, from (2) and (3), the local linear model is

$$M_n(x) = f(x_n) + \frac{1}{2}(x - x_n)[f'(x) + f'(x)].$$

Take the next iterative point as the root of $M_n(x)$, i.e. $M_n(x_{n+1}) = 0$, yields

$$M_n(x_{n+1}) = 0 = f(x_n) + \frac{1}{2}(x_{n+1} - x_n)[f'(x) + f'(x)] \Rightarrow x_{n+1} = x_n - \frac{2f(x_n)}{f'(x_n) + f'(x_{n+1})}, \qquad n = 0, 1, 2, \dots$$

Use Newton's iterative step to compute the $(n+1)^{\text{th}}$ iterate on the right-hand side. Thus, the resulting new scheme is

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)} \\ x_{n+1} &= x_n - \frac{2f(x_n)}{f'(x_n) + f'(y_n)}, \qquad n = 0, 1, 2, \dots \end{aligned} \qquad (4)$$

Obviously, this is an implicit method. This method converges of order three.

Also in **2000**, ***Wu*** and ***Wu***, proposed a new quadratic convergence iteration formulae with one parameter as follows:

$$x_{n+1} = x_n - \frac{f^2(x_n)}{\mu f^2(x_n) + f(x_n + f(x_n)) - f(x_n)}, \quad n = 0, 1, 2, \ldots$$

In **2001**, ***Gutierrez*** and ***Hernandez***, gives acceleration for Newton's method and obtained a new third order method named by Supper-Halley's method as follows:

Let $f:[a,b] \subseteq R \to R$ be a function satisfying $f'(t) < 0$, $f''(t) > 0$ for $t \in [a,b]$ and $f(a) > 0 > f(b)$. In this situation the sequence defined by

$$t_{n+1} = t_n - \frac{f(t_n)}{f'(t_n)}, \quad t_0 = a.$$

converges to t^* the only solution of $f(t)$ in [a,b].

Let g be another function satisfying the same conditions as f in [a,b], with g(t*)=0, and let $s_{n+1} = s_n - \frac{g(s_n)}{g'(s_n)}$, $s_0 = t_0$..Taking $g(t) = f'(t^*)(t - t^*)$ we obviously obtain a sequence that converges to t^* faster than $\{t_n\}$. The problem is that t^* is unknown. Instead of $f'(t^*)(t - t^*)$ take its Taylor approximation

$$g(t) = f(t) - \frac{f''(t^*)}{2}(t - t^*)^2.$$

So,

$$g(t_n) \approx f(t_n) - \frac{f''(t_n)}{2}(t_n - t_{n+1})^2 \text{ and } \quad g'(t_n) \approx f'(t_n) - f''(t_n)(t_n - t_{n+1}).$$

Put these values in the formula $s_{n+1} = t_n - \frac{g(t_n)}{g'(t_n)}$, we obtain

$$s_{n+1} = t_n - \left(1 + \frac{\widetilde{K}_{f(t_n)}}{2(1 - \widetilde{K}_{f(t_n)})}\right)\frac{f(t_n)}{f'(t_n)} \text{ where } \widetilde{K}_{f(t_n)} = \frac{f''(t_n)f(t_n)}{f'(t_n)^2}.$$

Also, in **2001**, ***Sebah*** and ***Gourdon***, derived two methods which are given by the following iteration formulas:

The first iteration formula: $x_{n+1} = x_n + (p+1)\left(\frac{(1/f)^{(p)}}{(1/f)^{(p+1)}}\right)_{x_n}$,

where p is an integer and $(1/f)^{(p)}$ is the derivative of order p of the inverse of the function f. This iteration has convergence of order $(p+2)$.

The second iteration formula:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}\left(1 + \frac{f(x_n)f''(x_n)}{2!(f'(x_n))^2} + \frac{(f'(x_n))^2\left(3(f''(x_n))^2 - f'(x_n)f'''(x_n)\right)}{3!(f'(x_n))^4}\right)$$

This iteration method convergence of order three.

In **2003**, ***Abbasbandy***, derived the following two formulas

$$x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}-\frac{f^2(x_n)f''(x_n)}{2(f'(x_n))^3}-\frac{f^3(x_n)(f''(x_n))^2}{2(f'(x_n))^5}$$

$$x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}-\frac{f^2(x_n)f''(x_n)}{2(f'(x_n))^3}-\frac{f^3(x_n)f'''(x_n)}{6(f'(x_n))^4}$$

by using Adomian decomposition method which are improvements of Newton-Raphson method.

In a similar manner of (4), ***Ozban***, in **2004**, derived the following two formulas

$$x_{n+1}=x_n-\frac{f(x_n)\left(f'(x_n)+f'(z_{n+1})\right)}{2f'(x_n)f'(z_{n+1})},\quad n=0,1,\ \dots$$

and

$$x_{n+1}=x_n-\frac{f(x_n)}{f'\left(\frac{x_n+z_{n+1}}{2}\right)},\quad n=0,1,\ \dots \text{ where } z_{n+1}=x_n-\frac{f(x_n)}{f'\left(x_n\right)},\quad n=0,1,\ \dots$$

by changing the arithmetic mean of $f'(x_n)$ and $f'(x_{n+1})$ instead of $f'(x_n)$ in Newton's method by harmonic mean. These two methods converge cubically.

In **2005**, there are some new iterative methods for solving nonlinear equations. ***Knawar et. al*** derived new formula as follows:

Assume

$$x_1=x_0+h,\qquad |h|\le 1, \tag{5}$$

be the first approximation to the root of the nonlinear equation (1).

Consider the following auxiliary equation with a parameter p:

$$g(x) = p^2 (x - x_0)^2 f^2(x) - f^2(x) = 0, \tag{6}$$

where $p \in R$ and $|p| < \infty$. It is clear that any root of (1) is also a root of (6) and vice versa. If $x_1 = x_0 + h$ is a better approximation for the required root where x_0 be the known initial guess for the required root, then (6) gives

$$p^2 h^2 f^2(x_0 + h) - f^2(x_0 + h) = 0. \tag{7}$$

Expanding by the Taylor's series of order two and simplifying, we get

$$h = \frac{2f(x_0)f'(x_0) \pm \sqrt{4f^2(x_0)f^2(x_0) + 4f^2(x_0)\left(p^2 f^2(x_0) - f'^2(x_0)\right)}}{2[p^2 f^2(x_0) - f'^2(x_0)]}. \tag{8}$$

Retaining the terms upto $o(h^2)$ excluding the term containing second derivative. Rationalize the numerator of (8) to obtain the new formula

$$h = \frac{-f(x_0)}{[f'(x_0) \pm pf(x_0)]}. \tag{9}$$

Using (9) in (5), the general formula for successive approximation is given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n) \pm pf(x_n)}, \quad n = 0, 1, \ldots$$

The parameter p is chosen such that the corresponding function $pf(x_n)$ and $f'(x_n)$ have the same sign. If we let $p \to 0$, the Newton-Raphson method is obtained.

Also in **2005**, ***Chun***, by using Adomian decomposition method, derived the following formula which is converges of order four

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - 2\frac{f(z_{n+1})}{f'(x_n)} + \frac{f(z_{n+1})f'(z_{n+1})}{(f'(x_n))^2}, \quad \text{where } z_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1, \ldots$$

In **2006**, many authors have considered methods for solving the nonlinear equations. We review among them the famous methods as follows:

A new two-point root-finding technique for solving nonlinear equation based on Adomian decomposition method is proposed by ***Abu-Alshaikh*** and ***Sahin,*** and given by:

$$x_{n+1} = x_{n-1} - \frac{f(x_{n-1})(x_n - x_{n-1})}{f(x_n) + f'(x_{n-1})(x_n - x_{n-1})}, \ n = 1, 2, \ldots \text{ where } f \in C^2[a,b]$$

Also based on Adomian decomposition method they proposed two other methods

$$(1)\ x_{n+1} = x_{n-1} - \frac{(x_n - x_{n-1})f(x_{n-1})}{f(x_n) + (x_n - x_{n-1})f'(x_{n-1})} - \frac{1}{2}\frac{(x_n - x_{n-1})^3 f''(x_{n-1}) f^2(x_{n-1})}{\{f(x_n) + (x_n - x_{n-1})f'(x_{n-1})\}^3}, \ n = 1, 2, \ldots$$

$$(2)\ x_{n+1} = x_{n-1} - \frac{(x_n - x_{n-1})f(x_{n-1})}{f(x_n) + (x_n - x_{n-1})f'(x_{n-1})} - \frac{1}{2}\frac{(x_n - x_{n-1})^3 f''(x_{n-1}) f^2(x_{n-1})}{\{f(x_n) + (x_n - x_{n-1})f'(x_{n-1})\}^3} - \frac{1}{2}\frac{(x_n - x_{n-1})^3 (f''(x_{n-1}))^2 f^3(x_{n-1})}{\{f(x_n) + (x_n - x_{n-1})f'(x_{n-1})\}^5}, \ n = 1, 2, \ldots$$

New three-step iterative method for solving nonlinear equations is proposed by ***Noor*** and ***Noor***

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \qquad f'(x_n) \neq 0,$$

$$z_n = -\frac{f(y_n)}{f'(x_n)}.$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{f(y_n + z_n)}{f'(x_n)}, \; n = 0, 1, 2....$$

This method converges of order three and derived by using Adomian decomposition method.

Also, depending on the Adomian decomposition method ***Noor, et. al*** proposed two iterative methods:

(1) $$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{f^2(x_n)}{2(f'(x_n))^3} f''(x_n), \; n = 0, 1, 2....$$

(2)

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \qquad f'(x_n) \neq 0,$$

$$z_n = -\frac{(y_n - x_n)^2}{2f'(x_n)} f''(x_n).$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{(y_n + z_n - x_n)^2}{2f'(x_n)} f''(x_n), \; n = 0, 1, 2....$$

The second method converges of order three.

Chum and ***Ham*** proposed the following Newton-like iteration method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{f(x_{n+1}^*)}{f'(x_n)} \text{ where } x_{n+1}^* = x_n - \frac{f(x_n)}{f'(x_n)}, \; n = 0, 1, 2....$$

which is converges of order four and derived by using the homotopy perturbation method.

In a similar manner of **Knawar et al. (2005),** ***Noor*** *and* ***Ahmed*** derived the following iterative methods:

(1) $$x_{n+1} = x_n - \frac{2f(x_n)}{f'(x_n) \pm \sqrt{(f'(x_n))^2 + 4p^3 f^3(x_n)}}, \; n = 0, 1, ...$$

where sign should be chosen so as to make the denominator largest in magnitude.

(2) $$z_n = x_n - \frac{2f(x_n)}{f'(x_n) \pm \sqrt{(f'(x_n))^2 + 4p^3 f^3(x_n)}}$$

$$x_{n+1} = z_n - \frac{f(z_n)}{f'(z_n)}, \; n = 0, 1, 2....$$

where sign should be chosen so as to make the denominator largest in magnitude. This method converges of order four.

Also, in a similar manner of ***Noor*** *and* ***Ahmed*** **(2006),** ***Noor et.al*** derived the following two-step iterative methods:

(1) $$z_n = x_n - \frac{2f(x_n)}{f'(x_n) \pm \sqrt{(f'(x_n))^2 + 4p^2 f^2(x_n)}},$$

$$x_{n+1} = z_n - \frac{f(z_n)}{f'(z_n)}, \; n = 0, 1, 2....$$

where sign should be chosen so as to make the denominator largest in magnitude. Here $p \in R$ is chosen so that $f(x_n)$ and p have the same sign.

(2) $$z_n = x_n - \alpha \frac{2f(x_n)}{f'(x_n) \pm \sqrt{(f'(x_n))^2 + 4p^2 f^2(x_n)}},$$

$$x_{n+1} = x_n + 4(z_n - x_n)\frac{f(x_n)}{3f(x_n) - 2f(z_n)}, \; n = 0, 1, 2....$$

where sign should be chosen so as to make the denominator largest in magnitude. Here $p \in R$ is chosen so that $f(x_n)$ and p have the same sign and α controlling parameter.

The developments of numerical techniques for solving nonlinear algebraic equations in 2007 are continued and the following methods are presented:

Chun, derived many iterative methods as follows:

Method 1:

Classical Chebyshev-Halley methods which is an improvement of Newton-Raphson method are given by:

$$x_{n+1} = x_n - \left(1 + \frac{1}{2}\frac{t(x_n)}{1-\beta t(x_n)}\right) \tag{10}$$

where

$$t(x_n) = \frac{f''(x_n) f(x_n)}{f'^2(x_n)}. \tag{11}$$

Let

$$y_n = x_n - \theta \frac{f(x_n) f'(x_n)}{f'^2(x_n) - \lambda f(x_n)}, \; n = 0,1,\ldots \tag{12}$$

where θ and λ are real parameters. Observe that if $\theta = 1$ and $\lambda = 0$ then (12) reduces to Newton-Raphson method. Considering the following approximation to $f''(x_n)$ in (11)

$$f''(x_n) \approx \frac{f'(x_n) - f'(y_n)}{x_n - y_n} = \frac{(f'(x_n) - f'(y_n))[f'^2(x_n) - \lambda f(x_n)]}{\theta f(x_n) f'(x_n)}.$$

Substitute the above approximation of $f''(x_n)$ in (11) yields a new three-parameter family of iterative methods free from second derivative and requiring three functional evaluations

$$x_{n+1} = x_n - \left(1 + \frac{1}{2}\frac{\left[f'(x_n) - f'(y_n)\right]\left[f'^2(x_n) - \lambda f(x_n)\right]}{\theta f'^3(x_n) - \beta\left[f'(x_n) - f'(y_n)\right]\left[f'^2(x_n) - \lambda f(x_n)\right]}\right)\frac{f(x_n)}{f'(x_n)},$$

where $y_n = x_n - \theta \frac{f(x_n) f'(x_n)}{f'^2(x_n) - \lambda f(x_n)}$, $n = 0,1,\ldots$, $\theta, \beta, \lambda \in R$.

For $\theta = \frac{2}{3}$ and $\beta = 1$ this method converges of order four.

Also he obtained the following iteration forms:

Method 2: $x_{n+1} = x_n - \left(1 + \frac{1}{2} \frac{K_\alpha(x_n)}{1 - \beta K_\alpha(x_n)}\right) \frac{f(x_n)}{f'(x_n)}$,

where $\beta, \alpha \in R$, $K_\alpha(x_n) = \frac{2 f(x_n) f(w_n)(1 + \alpha f'(x_n))}{f^2(x_n) + \alpha f'^2(x_n)[f(w_n) - f(x_n)]^2}$ and $w_n = x_n - \frac{f(x_n)}{f'(x_n)}$, $n = 0,1,\ldots$.

This method is a method of two-parameter family free from second derivative and converges of order three.

Method 3: $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{1}{2\theta} \frac{x_n - \xi(x_n)}{x_n - \theta(x_n)} \left(1 - \frac{f'(y_n)}{f'(x_n)}\right) \frac{f(x_n)}{f'(x_n)}$,

where $\theta(x) = x - \frac{f(x)}{f'(x)}$, $y_n = x_n - \theta[x_n - \theta(x_n)]$, $n = 0,1,\ldots$ and $\xi(x)$ any iteration function of order at least two. This method converges of order three.

Method 4:

$$z_n = x_n - \frac{f(x_n)}{f'(x_n)},$$

$$x_{n+1} = z_n - \frac{f(z_n) f'(x_n)}{f'^2(x_n) + (\lambda - 2\mu x_n) f(x_n) + \mu f^2(x_n)}, \quad n = 0, 1, \ldots$$

$\lambda, \mu \in R$. This method converges of order three..

Method 5:

$$x_{n+1} = x_n + (1+\beta)\frac{f(x_n) + f(y_n)}{f'(x_n)} - 2\frac{f^2(x_n)}{f'(x_n)[f(x_n) + f(y_n)]} - \beta\left[\frac{f(x_n)}{f'(x_n)} + \frac{f'(x_n) f(y_n)}{f^2(x_n) + f'^2(x_n)}\right],$$

where $y_n = x_n - \dfrac{f(x_n)}{f'(x_n)}$, and $\beta \in R$.

This method converges of order four.

For $\beta = -1$, this method reduced to the following fourth-order method

$$x_{n+1} = x_n - 2\frac{f^2(x_n)}{f'(x_n)[f(x_n) + f(y_n)]} + \left[\frac{f(x_n)}{f'(x_n)} + \frac{f'(x_n) f(y_n)}{f^2(x_n) + f'^2(x_n)}\right].$$

For $\beta = -2$, this method reduced to the following fourth-order method

$$x_{n+1} = x_n - \frac{f(x_n) + f(y_n)}{f'(x_n)} - 2\frac{f^2(x_n)}{f'(x_n)[f(x_n) + f(y_n)]} + 2\left[\frac{f(x_n)}{f'(x_n)} + \frac{f'(x_n) f(y_n)}{f^2(x_n) + f'^2(x_n)}\right],$$

Chun and ***Ham***, present one-parameter fourth-order family of iterative methods for solving nonlinear equations as follows:

Kou et al.'s third-order method defined by

$$x_{n+1} = x_n - \frac{f(x_n)}{2}\left[\frac{1}{f'(x_n)} + \frac{1}{2f'(z_n) - f'(x_n)}\right], \quad (13)$$

where

$$z_n = x_n - \frac{f(x_n)}{2f'(x_n)}. \quad (14)$$

To derive the proposed methods, firstly use the Taylor approximation combined with (14):

$$f'(z_n) \approx f'(x_n) + f''(x_n)(z_n - x_n) = f'(x_n) - \frac{1}{2}\frac{f(x_n)f''(x_n)}{f'(x_n)}. \quad (15)$$

As an approximation of $f'(z_n)$ in (13) to obtain the iterative scheme function and containing the second derivative of *f*:

$$x_{n+1} = x_n - \frac{1}{2}\frac{2f'^2(x_n) - f(x_n)f''(x_n)}{f'^2(x_n) - f(x_n)f''(x_n)}\frac{f(x_n)}{f'(x_n)}. \quad (16)$$

Approximate $f''(x_n)$ in (16) back by the use only of the function and its derivative . To do that, let

$$y_n = x_n - \theta\frac{f(x_n)f'(x_n)}{f'^2(x_n) - \lambda f(x_n)}, \quad (17)$$

where θ and λ are real parameters. The iteration (17) with $\theta = 1$ has been derived by (***Chun, 2006***) that is converges quadratically. Approximate $f''(x_n)$ by:

$$f''(x_n) \approx \frac{2[f(y_n) - f(x_n)][f'^2(x_n) - \lambda f(x_n)]^2}{\theta^2 f^2(x_n) f'^2(x_n)} + \frac{2[f'^2(x_n) - \lambda f(x_n)]^2}{\theta f(x_n)}, \qquad (18)$$

which can be derived from Taylor expansion of $f(y_n)$ about x_n

$$f(y_n) \approx f(x_n) + f'(x_n)(y_n - x_n) + \frac{1}{2} f''(x_n)(y_n - x_n)^2$$

$$= f(x_n) - \theta \frac{f(x_n) f'^2(x_n)}{f'^2(x_n) - \lambda f(x_n)} + \frac{1}{2}\theta^2 \frac{f^2(x_n) f'^2(x_n) f''(x_n)}{[f'^2(x_n) - \lambda f(x_n)]^2}. \qquad (19)$$

By using (18) in (16), he get the following two-parameter family of methods:

$$x_{n+1} = x_n - \left[\frac{(f(y_n) - f(x_n))L_n^2 + \theta f(x_n) f'^2(x_n)((1-\theta) f'^2(x_n) - \lambda f(x_n))}{2(f(y_n) - f(x_n))L_n^2 + \theta f(x_n) f'^2(x_n)((2-\theta) f'^2(x_n) - 2\lambda f(x_n))} \right] \frac{f(x_n)}{f'(x_n)},$$

where $\lambda,\ \theta \in R,\ L_n = f'^2(x_n) - \lambda f(x_n)$ and $y_n = x_n - \theta \frac{f(x_n) f'(x_n)}{f'^2(x_n) + \lambda f(x_n)},\ n = 0, 1, \ldots.$

This method converges of order four if $\theta = 1$.

Also, ***Ham*** and ***Chun*** derived the fifth-order convergence iteration method as follows:

Let us consider the iteration scheme in the form

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \qquad (20)$$

$$x_{n+1} = y_n - \frac{Af'(y_n) + Bf'(x_n)}{Cf'(y_n) + Df'(x_n)} \times \frac{f(y_n)}{f'(x_n)}, \quad n = 0,\ 1,\ \ldots. \qquad (21)$$

and they showed that the method defined by (20) and (21) is of fifth order if $A + B = C + D,\ C = B + A,\ B = 3A,\ \ C + D \neq 0$. For $A = 1,\ B = 3, C = 5$ and $D = -1$ yields

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)},$$

$$x_{n+1} = y_n - \frac{f'(y_n) + 3f'(x_n)}{5f'(y_n) - f'(x_n)} \times \frac{f(y_n)}{f'(x_n)}, \quad n = 0,\ 1,\ \ldots.$$

Feng and ***He***, derived the following iterative methods of order two and three respectively by using homotopy perturbation method:

(1) $x_{n+1} = x_n - \dfrac{2hf(x_n)}{f(x_n + h) - f(x_n - h)}, \quad n = 0,\ 1,\ \ldots$

(2) $x_{n+1} = x_n - \dfrac{2hf(x_n)}{f(x_n + h) - f(x_n - h)} - \dfrac{4hf^2(x_n)\left[f(x_n + h) + f(x_n - h) - 2f(x_n)\right]}{\left[f(x_n + h) - f(x_n - h)\right]^3},$

$n = 0,\ 1,\ \ldots$

where $h = \alpha f(x_n)$ and $\alpha \neq 0$ is parameter.

Sharma and ***Guha***, derived the following multipoint iterative method,

$$z_n = x_n - \frac{f(x_n)}{f'(x_n)},$$
$$v_n = z_n - \frac{f(z_n)}{f'(z_n)} \times \frac{f(x_n)}{f(x_n) - 2f(z_n)}$$
$$x_{n+1} = v_n - \frac{f(v_n)}{f'(v_n)} \times \frac{f(x_n) + af(z_n)}{f(x_n) + bf(z_n)}, \; n = 0,\; 1, \ldots$$

named by modified Ostrowski method and converges of order six if $b=a-2$. For a=0 and b=-2, the iterative formula is as follows:

$$z_n = x_n - \frac{f(x_n)}{f'(x_n)},$$
$$v_n = z_n - \frac{f(z_n)}{f'(z_n)} \times \frac{f(x_n)}{f(x_n) - 2f(z_n)}$$
$$x_{n+1} = v_n - \frac{f(v_n)}{f'(v_n)} \times \frac{f(x_n)}{f(x_n) - 2f(z_n)}, \; n = 0,\; 1, \ldots$$

Ostrowski method given by

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)},$$
$$x_{n+1} = y_n - \frac{f(x_n)}{f(x_n) - 2f(y_n)} \times \frac{f(y_n)}{f'(x_n)}$$

which are improvements of Newton’s method, the order increases by at least two.

Grau et .al develop the sixth-order variant of Ostrowski's method which is given by

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)},$$

$$z_n = y_n - \frac{f(x_n)}{f(x_n) - 2f(y_n)} \times \frac{f(y_n)}{f'(x_n)}$$

$$x_{n+1} = z_n - \frac{f(x_n)}{f(x_n) - 2f(y_n)} \times \frac{f(z_n)}{f'(x_n)}$$

Another sixth-order family of modified Ostrowski's method was considered by **Sharma et al.**

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)},$$

$$z_n = y_n - \frac{f(x_n)}{f(x_n) - 2f(y_n)} \times \frac{f(y_n)}{f'(x_n)}$$

$$x_{n+1} = z_n - \frac{f(x_n) + (\beta - 2) f(y_n)}{f(x_n) - 2f(y_n)} \times \frac{f(z_n)}{f'(x_n)}$$ where $\beta \in R$.

Also ***Chun*** and ***Ham***, presented another sixth-order variant of Ostrowski's method

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)},$$

$$z_n = y_n - \frac{f(x_n)}{f(x_n) - 2f(y_n)} \times \frac{f(y_n)}{f'(x_n)}$$

$x_{n+1} = z_n - H(u_n) \times \frac{f(z_n)}{f'(x_n)}$ where $u_n = \frac{f(y_n)}{f'(x_n)}$ and $H(t)$ represents a real-valued function satisfy the properties $H(0) = 1, \;\; H'(0) = 2.$

Noor and ***Noor***, by using Adomian decomposition method derived the following five iterative methods:

(1) $y_n = x_n - \frac{f(x_n)}{f'(x_n)},$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{(y_n - x_n)^2}{2f'(x_n)} f''(x_n), \;\; n = 0, 1, \ldots$$

(2) $y_n = x_n - \frac{f(x_n)}{f'(x_n)},$

$$z_n = -\frac{(y_n - x_n)^2}{2f'(x_n)} f''(x_n),$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{(y_n - x_n)^2}{2f'(x_n)} f''(x_n) - \frac{(y_n + z_n - x_n)^2}{2f'(x_n)} f''(x_n), \;\; n = 0, 1, \ldots$$

Converge of order three.

(3) $$y_n = x_n - \frac{f(x_n)}{f'(x_n)},$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{f(y_n)}{f'(x_n)}, \quad n = 0, 1, \ldots$$

(4) $$y_n = x_n - \frac{f(x_n)}{f'(x_n)},$$

$$z_n = -\frac{f(y_n)}{f'(y_n)},$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{f(y_n)}{f'(x_n)} - \frac{f(y_n + z_n)}{f'(x_n)}, \quad n = 0, 1, \ldots$$

Converge of order four

(5) $$y_n = x_n - \frac{f(x_n)}{f'(x_n)},$$

$$z_n = y_n - \frac{f(y_n)}{f'(y_n)},$$

$$x_{n+1} = z_n - \frac{f(z_n)}{f'(z_n)}, \quad n = 0, 1, \ldots$$

Also, in a similar manner of the above iterative method by ***Noor*** and ***Noor***, ***Noor*** and ***Gupta*** proposed an algorithm of order four which is given by

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)},$$

$$x_{n+1} = x_n - \frac{f(y_n)}{f'(y_n)} - \frac{1}{2}\left[\frac{f(y_n)}{f'(y_n)}\right]^2 \left[\frac{f(x_n)}{f'(x_n)}\right]\left[\frac{f'(x_n)+f'(y_n)}{f'(y_n)}\right], \quad n = 0, 1, \ldots$$

Kou and ***Wang***, presented a variant of Ostrowski's method with order of convergence seven and given by:

$$x_{n+1} = z_n - \left[(1+H_2(x_n, y_n))^2 + H_\alpha(y_n, z_n)\right]\left[\frac{f(z_n)}{f'(x_n)}\right], \quad n = 0, 1, \ldots$$

where

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \quad H_2(x_n, y_n) = \frac{f(y_n)}{f(x_n) - 2f(y_n)}, \quad H_\alpha(y_n, z_n) = \frac{f(z_n)}{f(y_n) - \alpha f(z_n)}$$ and $$z_n = x_n - (1+H_2(x_n, y_n))\frac{f(x_n)}{f'(x_n)}.$$

The most important iterative methods for solving nonlinear equations in 2008 are the following:

Bi et. al derived a seventh-order iterative method for solving non-linear equation as as follows:

A one-parameter family of ourth-order methods, which is derived by King (1973) developed is given by:

$$\left.\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\ x_{n+1} &= y_n - \frac{f(x_n)+\beta f(y_n)}{f'(x_n)+(\beta-2)f(y_n)}\,\frac{f(y_n)}{f'(x_n)}, \end{aligned}\right\} \qquad (22)$$

where $\beta \in R$ is a constant.

Now, combining (2) with Newton's method, yields

$$\left.\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n &= y_n - \frac{f(x_n)+\beta f(y_n)}{f'(x_n)+(\beta-2)f(y_n)}\,\frac{f(y_n)}{f'(x_n)}, \\ x_{n+1} &= z_n - \frac{f(z_n)}{f'(z_n)} \end{aligned}\right\} \qquad (23)$$

where $\beta \in R$ is a constant.

Using Taylor expansion, $f(z_n)$ and $f'(z_n)$ in (23) can be approximated by

$$f(z_n) \approx f(y_n) + f'(y_n)(z_n - y_n) + \frac{1}{2} f''(y_n)(z_n - y_n)^2, \qquad (24)$$

$$f'(z_n) \approx f'(y_n) + f''(y_n)(z_n - y_n). \qquad (25)$$

Then

$$f'(z_n) \approx \frac{f(z_n) - f(y_n)}{z_n - y_n} + \frac{1}{2} f''(y_n)(z_n - y_n). = f[z_n, y_n] + \frac{1}{2} f''(y_n)(z_n - y_n) \qquad (26)$$

In order to avoid the computation of the second order derivative, approximate $f''(y_n)$ as follows:

$$f''(y_n) \approx 2f[z_n, x_n, x_n] = \frac{2(f[z_n, x_n] - f'(x_n))}{z_n - x_n} \tag{27}$$

where z_n and x_n are sufficiently close to y_n, when n is a sufficiently big integer. Substituting (27) into (26) and replace $f'(z_n)$ with approximation in (26), a new family of methods by (23), (26) and (27) can be constructed as follows:

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)},$$

$$z_n = y_n - \frac{f(x_n) + \beta f(y_n)}{f'(x_n) + (\beta - 2) f(y_n)} \frac{f(y_n)}{f'(x_n)},$$

$$x_{n+1} = z_n - \frac{f(z_n)}{f[z_n, y_n] + f[z_n, x_n, x_n](z_n - y_n)}, \quad n = 0, 1, \ldots$$

Basu, derive the fourth-order iterative method

$$x_{n+1} = x_n - \frac{f'\left(x_n - \frac{2}{3}\frac{f(x_n)}{f'(x_n)}\right) f(x_n)}{\frac{3}{16}\left\{f'\left(x_n - \frac{2}{3}\frac{f(x_n)}{f'(x_n)}\right)\right\}^2 + \frac{11}{8} f'\left(x_n - \frac{2}{3}\frac{f(x_n)}{f'(x_n)}\right) f'(x_n) - \frac{9}{16}[f'(x_n)]^2}, \quad n = 0, 1, \ldots$$

Depending on the Chebyshev method for simple root

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}\left[1 + \frac{1}{2}\frac{f(x_n)}{f'(x_n)}\frac{f''(x_n)}{f'(x_n)}\right], \quad n = 0, 1, \ldots$$

Neta, proposed the following two-parameter iterative method

$$x_{n+1} = x_n - \alpha\frac{f(x_n)}{f'(x_n)}\left[1 + \beta\frac{f(x_n)}{f'(x_n)}\frac{f''(x_n)}{f'(x_n)}\right], \quad n = 0, 1, \ldots$$

which converges of order two if $\beta = \frac{3}{2}\frac{3-\alpha}{\alpha}$ for $\alpha = 1$ or $\alpha = \frac{9}{7}$.

Depending on the two-step Newton iteration method of order four which is derived by ***Turab*** (1977)

$$w_n = x_n - \frac{f(x_n)}{f'(x_n)},$$

$$x_{n+1} = w_n - \frac{f(w_n)}{f'(w_n)}, \quad n = 0, 1, \ldots$$

Chun, derived another fourth-order iterative method which is given by:

$$w_n = x_n - \frac{f(x_n)}{f'(x_n)},$$

$x_{n+1} = w_n - \frac{f(w_n)}{f'(x_n)h(u_n)}, \quad n = 0, 1, \ldots$ where $u_n = \frac{f(w_n)}{f(x_n)}$, and $h(t)$ a real valued function satisfies the following conditions: $h(0) = 1$, $h'(0) = -2$ and $|h''(0)| < \infty$.

Parhi and ***Gupta***, depending on the third-order method developed by ***Weerakoon*** and ***Fernando*****(2000)**

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)},$$

$$z_n = x_n - \frac{2f(x_n)}{f'(x_n) + f'(y_n)},$$

$$x_{n+1} = z_n - \frac{f(z_n)}{f'(z_n)}, \quad n = 0, 1, \ldots$$

and using the linear interpolation, presented a six-order method

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)},$$

$$z_n = x_n - \frac{2f(x_n)}{f'(x_n) + f'(y_n)},$$

$$x_{n+1} = z_n - \frac{f(z_n)}{f'(z_n)} \frac{f'(x_n) + f'(y_n)}{3f'(y_n) - f'(x_n)}, n = 0, 1, \ldots$$

Aziz, together with me, and using Adomian decomposition method derived the following iterative method of order four for solving nonlinear equations

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)},$$

$$z_n = -\left[\frac{(y_n - x_n)^2}{2!} f''(x_n) + \frac{(y_n - x_n)^3}{3!} f'''(x_n) + \frac{(y_n - x_n)^4}{4!} f^{(4)}(x_n)\right] \frac{1}{f'(x_n)},$$

$$x_{n+1} = y_n - \left[\frac{(y_n + z_n - x_n)^2}{2!} f''(x_n) + \frac{(y_n + z_n - x_n)^3}{3!} f'''(x_n) + \right.$$
$$\left. \frac{(y_n + z_n - x_n)^4}{4!} f^{(4)}(x_n) \right] \frac{1}{f'(x_n)}, \quad n = 0, 1, \ldots$$

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Finally, **Shno together with me derived eight new iterative methods in (2009) for solving nonlinear equations:**

The first iterative method is derived as follows:

There exists a modification of Newton's method with third-order convergence due to ***Porta*** and ***Pták*** (1994) defined by

$$x_{n+1} = x_n - \frac{f(x_n) + f(x_n - \frac{f(x_n)}{f'(x_n)})}{f'(x_n)}. \tag{28}$$

Recently, ***Zhou*** (2008) proposed a new third-order method, which is defined by:

$$x_{n+1} = x_n - \frac{f^2(x_n) - 2f(x_n)f(x_n - \frac{f(x_n)}{f'(x_n)})}{f(x_n)f'(x_n) - 3f'(x_n)f(x_n - \frac{f(x_n)}{f'(x_n)})}. \quad (29)$$

We presented a new fourth-order method based on the combination of formula (28) and (29). Now, we consider the linear combination of formula (28) and (29) which produces a class of high-order iterative method as follows:

$$x_{n+1} = x_n - \beta \frac{f(x_n) + f(y_n)}{f'(x_n)} - (1-\beta)\frac{f^2(x_n) - 2f(x_n)f(y_n)}{f(x_n)f'(x_n) - 3f'(x_n)f(y_n)}, \quad (30)$$

where $\beta \in R$, and

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)} \text{ for } n = 0,\ 1,\ \ldots\ . \quad (31)$$

Obviously, for $\beta = 1$, formula (30) reduces to (28), while we have (29) for $\beta = 0$.

Hereunder, we consider the convergence analysis of the formula (30) by the following theorem:

Theorem 1. Assume that the function $f(x)$ is sufficiently smooth in a neighborhood of its root α with $f'(\alpha) \neq 0$. Then the iterative method defined by (30) convergence of order four to α in a neighborhood of α if $\beta = \frac{1}{3}$.

Proof. By using Taylor's expansion, we have

$$f(x_n) = f'(\alpha)[e_n + c_2 e_n^2 + c_3 e_n^3 + c_4 e_n^4 + O(e_n^5)], \quad (32)$$

$$f'(x_n) = f'(\alpha)[1 + 2c_2 e_n + 3c_3 e_n^2 + 4c_4 e_n^3 + 5c_5 e_n^4 + O(e_n^5)], \quad (33)$$

where $c_j = \frac{f^{(j)}(\alpha)}{j!\, f'(\alpha)}$ for j=2, 3… and $e_n = x_n - \alpha$.

Dividing (32) by (33), gives us

$$\frac{f(x)}{f'(x)} = e_n - c_2 e_n^2 + (2c_2^2 - 2c_3)e_n^3 + (7c_2 c_3 - 4c_2^3 - 3c_4)e_n^4 + O(e_n^5). \quad (34)$$

Subsequently, from (31) and (34), we get

$$y_n = \alpha + c_2 e_n^2 + (2c_3 - 2c_2^2)e_n^3 + (4c_2^3 + 3c_4 - 7c_2 c_3)e_n^4 + O(e_n^5). \quad (35)$$

Also, by Taylor series, we have

$$f(y_n) = f'(\alpha)[2c_2 e_n^2 + (2c_3 - 2c_2^2)e_n^3 + (5c_2^3 + 3c_4 - 7c_2 c_3)e_n^4 + O(e_n^5)]. \quad (36)$$

So, from (32) and (36)

$$f(x_n) + f(y_n) = f'(\alpha)[e_n + 2c_2 e_n^2 + (2c_3 - 2c_2^2)e_n^3 + (5c_2^3 + 3c_4 - 7c_2 c_3)e_n^4 + O(e_n^5)]. \quad (37)$$

From (37) and (33) we have

$$\frac{f(x_n) + f(y_n)}{f'(x_n)} = e_n - 2c_2^2 e_n^3 - (7c_2 c_3 - 9c_2^3)e_n^4 + + O(e_n^5) \ . \quad (38)$$

Also, from (32) and (36) we have

$$f(x_n) - 2f(y_n) = f'(\alpha)[e_n - 2c_2 e_n^2 + (4c_2^2 - 3c_3)e_n^3 + (14c_2 c_3 - 10c_2^3 - 5c_4)e_n^4 + + O(e_n^5)]. \quad (39)$$

and

$$f(x_n)-3f(y_n)=f'(\alpha)[e_n-2c_2e_n^2+(6c_2^2-5c_3)e_n^3+(21c_2c_3-15c_2^3-8c_4)e_n^4++O(e_n^5)]. \qquad (40)$$

Dividing (39) by (40), gives us

$$\frac{f(x_n)-2f(y_n)}{f(x_n)-3f(y_n)}=1+2c_2e_n+3c_3e_n^2+(3c_4+2c_2c_4-c_2^3)e_n^3+Oe_n^4. \qquad (41)$$

From (32), (33) and (36), we get

$$\frac{f^2(x_n)-2f(y_n)f(x_n)}{f(x_n)f'(x_n)-3f(y_n)f'(x_n)}=e_n+c_2^2e_n^3+(5c_2c_4-3c_2^3c_3)e_n^4+O(e_n^5). \qquad (42)$$

From (30), (38), (42) and $e_{n+1}=x_{n+1}-\alpha$, we get

$$e_{n+1}=(3\beta c_2^2-c_2^2)e_n^3+(12\beta c_2c_3-5c_2c_3-12\beta c_2^3+3c_2^3)e_n^4+O(e_n^5). \qquad (43)$$

From equation (43), we see that the method defined by (30) cubically convergent for any $(\beta\neq\frac{1}{3})\in R$. Furthermore, when $\beta=\frac{1}{3}$, the method has fourth-order of convergence because $3\beta c_2^2-c_2^2=0$ for $\beta=\frac{1}{3}$. ♦

Replacing β in (30) by $\frac{1}{3}$, we obtain the following algorithm with fourth-order convergence:

Algorithm 1:

Step 1: Let *n*=0. For given x_0 calculate x_1, $x_2 \dots$ such that

$$x_{n+1} = x_n - \frac{f^2(x_n) - 2f(x_n)f(y_n) - f^2(y_n)}{f'(x_n)(f(x_n) - 3f(y_n))},$$

where $y_n = x_n - \dfrac{f(x_n)}{f'(x_n)}$.

Step 2: For a give $\varepsilon > 0$, if $|f(x_n)| < \varepsilon$, then stop.

Step 3: Set $n = n+1$ and go to Step 1.

Another iterative methods presented by Saeed and Shno and derived as follows:

The supper-Halley method defined by

$$x_{n+1} = x_n - \left(1 + \frac{1}{2}\frac{\tilde{K}_{f(x_n)}}{1 - \tilde{K}_{f(x_n)}}\right)\frac{f(x_n)}{f'(x_n)}, \tag{44}$$

where

$$\tilde{K}_{f(x_n)} = \frac{f''(x_n)f(x_n)}{f'(x_n)^2}. \tag{45}$$

Another improvement of Newton's method proposed by ***Kou*** and ***Wang*** (2007), which is defined by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n) + \lambda f(x_n)} , \tag{46}$$

where $\lambda \in R$, $0 < |\lambda| < +\infty$ and λ is chosen such that the corresponding function values $\lambda f(x_n)$ and $f'(x_n)$ have the same signs. This method converges quadratically under the condition $f'(x_n) + \lambda f(x_n) = 0$, while $f'(x_n) \neq 0$ in some points is permitted.

To derive this method, let:

$$y_n = x_n - \theta \frac{f(x_n)}{f'(x_n) + f(x_n)} , \tag{47}$$

where θ a nonzero real parameter, and we approximate $f''(x_n)$ in equation (45) by $f''(y_n)$.

By combining equations (47) and (44), we have the following two-step iterative method which is an improvement of the supper-Halley method:

$$\left.\begin{aligned} y_n &= x_n - \theta \frac{f(x_n)}{f'(x_n) + f(x_n)}, \quad \theta \in R \\ x_{n+1} &= x_n - \left(1 + \frac{1}{2} \frac{K_{f(x_n)}}{1 - K_{f(x_n)}}\right) \frac{f(x_n)}{f'(x_n)} \end{aligned}\right\}, \tag{48}$$

where $K_{f(x_n)} = \dfrac{f''(y_n)f(x_n)}{f'(x_n)^2}$.

Also, to develop another method which is three-step iterative method, we combine equation (48) with equation (1) as follows:

$$\left.\begin{aligned} y_n &= x_n - \theta \frac{f(x_n)}{f'(x_n) + f(x_n)}, \quad \theta \in R \\ z_n &= x_n - \left(1 + \frac{1}{2}\frac{K_{f(x_n)}}{1 - K_{f(x_n)}}\right)\frac{f(x_n)}{f'(x_n)} \\ x_{n+1} &= z_n - \frac{f(z_n)}{f'(z_n)} \end{aligned}\right\} \tag{49}$$

where $K_{f(x_n)} = \dfrac{f''(y_n)f(x_n)}{f'(x_n)^2}$.

In equation (49), approximate $f'(z_n)$ by the following:

$$f'(z_n) = 3\frac{f(z_n) - f(x_n)}{(z_n - x_n)} - 2f'(x_n) - \frac{1}{2}f''(x_n)(z_n - x_n). \tag{50}$$

Now, by replacing $f'(z_n)$ in equation (49) by (50), we can suggest another new three-step iterative method which is also an improvement of the supper-Halley method as follows:

$$\left.\begin{aligned} y_n &= x_n - \theta \frac{f(x_n)}{f'(x_n) + f(x_n)}, \quad \theta \in R \\ z_n &= x_n - \left(1 + \frac{1}{2} \frac{K_{f(x_n)}}{1 - K_{f(x_n)}}\right) \frac{f(x_n)}{f'(x_n)}, \\ x_{n+1} &= z_n - \frac{f(z_n)}{A}, \end{aligned}\right\} \qquad (51)$$

where $A = 3\frac{f(z_n) - f(x_n)}{(z_n - x_n)} - 2f'(x_n) - \frac{1}{2} f''(x_n)(z_n - x_n)$ and $K_{f(x_n)} = \frac{f''(y_n) f(x_n)}{f'(x_n)^2}$.

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We study the order of convergence of equation (50) and equation (51) by the following theorems:

Theorem. Assume that the function $f : D \subset R \to R$ has a simple root $\alpha \in D$ and let $f(x)$ be sufficiently smooth function in the neighborhood of the root α. Then the iterative method (48) has fourth-order convergence if $\theta = \frac{1}{3}$. ♦

By setting $\theta = \frac{1}{3}$, we rewrite equation (48) as the following Algorithm for solving nonlinear equation $f(x) = 0$ which has fourth-order convergence:

Algorithm 2:

Step 1: For given x_0 calculate $x_1,\ x_2\ldots$ such that

$$x_{n+1} = x_n - \left(1 + \frac{1}{2}\frac{K_{f(x_n)}}{1 - K_{f(x_n)}}\right)\frac{f(x_n)}{f'(x_n)}\text{ , where }\quad K_{f(x_n)} = \frac{f''\left(x_n - \dfrac{f(x_n)}{3\left(f'(x_n) + f(x_n)\right)}\right) f\left(x_n\right)}{f'\left(x_n\right)^2}.$$

Step 2: For a give $\varepsilon > 0$, if $|f(x_{n+1})| < \varepsilon$, then stop.

Step 3: Set $n = n + 1$ and go to Step 1.

Theorem. Assume that the function $f : D \subset R \to R$ has a simple root $\alpha \in D$ and let $f(x)$ be sufficiently smooth function in the neighborhood of the root α. Then the iterative method (51) has seventh-order convergence if $\theta = \frac{1}{3}$. ♦

Algorithm 3:

Step 1: For given x_0 calculate $x_1,\ x_2\ldots$ such that

$$\left.\begin{aligned} z_n &= x_n - \left(1+\frac{1}{2}\,\frac{K_{f(x_n)}}{1-K_{f(x_n)}}\right)\frac{f(x_n)}{f'(x_n)}, \\ x_{n+1} &= z_n - \frac{f(z_n)}{A}, \end{aligned}\right\}$$

where $A = 3\dfrac{f(z_n)-f(x_n)}{(z_n-x_n)} - 2f'(x_n) - \dfrac{1}{2}f''(x_n)(z_n-x_n)$ and $K_{f(x_n)} = \dfrac{f''\left(x_n - \dfrac{f(x_n)}{3\left(f'(x_n)+f(x_n)\right)}\right)f(x_n)}{f'(x_n)^2}$.

Step 2: For a give $\varepsilon > 0$, if $|f(x_{n+1})| < \varepsilon$, then stop.

Step 3: Set $n = n+1$ and go to Step 1.

The following algorithms also derived by Saeed and Shno:

Algorithm 4:

Step 1: Here p is chosen so that $f(x_n)$ and p have the same sign. Let *n*=0. For given x_0 calculate x_1, $x_2\dots$ such that

$$x_{n+1} = x_n - \frac{2f(x_n)}{f'(x_n) \pm \sqrt{f'^2(x_n) + 4f(x_n)\left(p^3 f^2(x_n) - \frac{f''(x_n)}{2}\right)}},$$

where sign is chosen such as to make the denominator largest in magnitude.

Step 2: For a give $\varepsilon > 0$, if $|f(x_n)| < \varepsilon$, then stop.

Step 3: Set $n = n+1$ and go to Step 1.

This algorithm has at least third-order convergence.

Algorithm 5:

Step 1: Here p is chosen so that $f(x_n)$ and p have the same sign. Let *n*=0. For given x_0 calculate x_1, $x_2 \dots$ such that

$$y_n = x_n - \frac{2f(x_n)}{f'(x_n) \pm \sqrt{f'^2(x_n) + 4f(x_n)\left(p^3 f^2(x_n) - \frac{f''(x_n)}{2}\right)}},$$

$$x_{n+1} = y_n - \frac{f(y_n)}{f'(y_n)},$$

where sign is chosen such as to make the denominator largest in magnitude.

Step 2: For a give $\varepsilon > 0$, if $|f(x_n)| < \varepsilon$, then stop.

Step 3: Set $n = n+1$ and go to Step 1.

This algorithm has at least six-order convergence.

Algorithm 6:

Step 1: Let *n*=0. For given x_0 calculate x_1, $x_2 \ldots$ such that

$$\left.\begin{aligned} z_n &= x_n - \left(1 + \frac{K_{f(x_n)}}{2(1-K_{f(x_n)})}\right)\frac{f(x_n)}{f'(x_n)}, \\ x_{n+1} &= z_n - \frac{f(z_n)}{3\dfrac{f(z_n)-f(x_n)}{(z_n-x_n)} - 2f'(x_n) - \dfrac{1}{2}f''(x_n)(z_n-x_n)}, \end{aligned}\right\}$$

where $K_{f(x_n)} = \dfrac{f''\left(x_n - \dfrac{f(x_n)}{3\left(f'(x_n)+f(x_n)\right)}\right)f(x_n)}{f'(x_n)^2}$.

Step 2: For a give $\varepsilon > 0$, if $|f(x_{n+1})| < \varepsilon$, then stop.

Step 3: Set $n = n+1$ and go to Step 1.

This algorithm converges of order seven.

Also, for multiple roots she obtain the following algorithms of order three:

Algorithm 7:

Step 1: Let *n*=0. For given x_0, compute the approximate solution x_{n+1} by the following iterative schemes:

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)},$$

$$x_{n+1} = x_n - \frac{f^2(x_n)}{f'(x_n)\left(f(x_n) - \left(\frac{m}{m-1}\right)^{m-1} f(y_n)\right)}.$$

Step 2: For a give $\varepsilon > 0$, if $|f(x_{n+1})| < \varepsilon$, then stop.

Step 3: Set $n = n+1$ and go to Step 1.

Algorithm 8:

Step 1: Let *n*=0. For given x_0, compute the approximate solution x_{n+1} by the following iterative schemes:

$$x_{n+1} = x_n - \frac{f^2(x_n)}{f'(x_n)\left(\left(\frac{m^2+4m-1}{m(m+1)^2}\right)f(x_n) + \left(\frac{2^{m+1}}{(m+1)^2}\left[\frac{m-1}{m}\right]^{m-1}\right)f(y_n)\right)},$$

where $y_n = x_n - \left(\frac{m+1}{2}\right)\frac{f(x_n)}{f'(x_n)}$.

Step 2: For a give $\varepsilon > 0$, if $|f(x_{n+1})| < \varepsilon$, then stop.

Step 3: Set $n = n+1$ and go to Step 1.

Note: for details of algorithms (1)-(8), see MSc thesis by Shno, entitled "Improvements of some Iterative Methods for Solving Non-Linear Equations", Salahaddin University/Erbil, Iraq, 2009.