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Arrangements of Hyperplanes, Lower Central Series and Chen Lie Algebras

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I. Introduction

Given 2 topological spaces, a natural question is to know whether their fundamental groups are isomorphic or not. We will study the case of complements of complex arrangements of hyperplanes and we will give some partial and very incomplete answers.

Let me begin by giving some insights about arrangements of hyperplanes.

In its simplest manifestation, an arrangement is merely a finite collection of lines in the real plane whose the complement consists of a finite number of polygonal bounded and unbounded regions. Determining the numbers of these regions turns out to be purely combinatorial problem which can easily solve by a recursion whose solution is given by formulas involving only the number of lines and the number of lines through each intersection point.

These formulas generalize to collection of hyperplanes of $\mathbb{R}^l$ where the recursive formulas are satisfied by an evaluation of the characteristic polynomials forms of the (reverse-ordered) poset of intersections. The study of characteristic polynomials forms the back-bone of the combinatorial, and much of the algebraic theory of arrangements.

From the topological standpoint, a richer situation is presented by arrangements of complex hyperplanes (in $\mathbb{C}^l$ or $\mathbb{P}^l$). In this case, the complement is connected, and its topology, as reflected in the fundamental group or the cohomology ring for instance, is much more interesting.

The motivation and many of the applications of the topological theory arose initially from the connection with braids.

– *Fox and Neuwirth* (1962) : Let M_l be the complement of the diagonal hyperplanes in $\mathbb{C}^l$. Then $\pi_1(M_l) \simeq P_l$ (pure braid group with l-strands).
– *Fadell and Neuwith* (1962) : M_l is aspherical.
– *V.I. Arnold* (1969) : $H^*(M_l; \mathbb{C})$. It is the beginning of a very active period of researches in this area.
– *E. Brieskorn* (1970) : $H^*(M; \mathbb{C})$ for any arrangements where M is the complement.
– *P. Deligne* (1972) : M is aspherical where M is the complement of a complexified simplicial arrangement.
– *P. Orlik and L. Solomon* (1980) :
$H^*(M; \mathbb{Z}) \simeq A^*_{\mathbb{Z}}(\mathcal{A}) := E_{\mathbb{Z}}/\mathcal{J}$ (which is combinatorially defined)
and $P_M(t) = \sum_{X \in L(\mathcal{A})} \mu(X)(-t)^{codim(X)}$.
– *Rybnikov* (1994) : $\pi_1(M)$ is not combinatorially defined.

II. The Magnus theory

There is a very strong analogy between the theory of *groups* and the theory of *Lie algebras*.

Let G be an arbitrary group. Let G^{ab} be the *abelianization* of G, that is $G^{\mathrm{ab}} = G/[G,G]$ where $[G,G]$ is the subgroup of commutators.

If G is abelian, then $[G,G] = 0$ and $G^{\mathrm{ab}} = G$.

If G is *perfect*, then $[G,G] = G$ and $G^{\mathrm{ab}} = 0$.

Let us consider the *Lower Central Series* of G which is denoted by $(\Gamma_n G)_{n\geq 1}$ where :

- $\Gamma_1 G = G$
- $\Gamma_{n+1} G = [G, \Gamma_n G]$

Properties :

- $\Gamma_{n+1} G$ is a subgroup of $\Gamma_n G$
- $\Gamma_n G / \Gamma_{n+1} G$ is an abelian group which is finitely generated if G is finitely generated.
- $[\Gamma_m G, \Gamma_n G] \subset \Gamma_{m+n} G$.

Define

$$\mathrm{gr}_n G = \Gamma_n G / \Gamma_{n+1} G$$

which is an abelian group for any $n \geq 1$ and

$$\mathrm{gr} G = \bigoplus_{n \geq 1} \mathrm{gr}_n G$$

There is a natural structure of Lie algebra on gr$G$ over $\mathbb{Z}$ where the Lie bracket $[x, y]$ is induced from the group commutator

$$(x, y) = xyx^{-1}y^{-1}$$

Let denote $\phi_n(G) = \mathrm{rank}(\mathrm{gr}_n G)$. They are important numerical invariants of G. Although they may be very difficult to determine, many properties of the group G are reflected into properties of its associated Lie algebra gr$G$. Then a natural question is, given the group G, to determine the Lie algebra gr$G$ and to compute $\phi_n(G)$ for every n.

III. Braid Groups

The *Braid group with l strands* denoted B_l admits the following presentation :
– generators : $\sigma_1, \ldots, \sigma_{l-1}$
– relations : $\sigma_i \sigma_j = \sigma_j \sigma_i$ for $\mid i - j \mid > 1$ and
$\sigma_{i+1}\sigma_i\sigma_{i+1} = \sigma_i\sigma_{i+1}\sigma_i$ for $i \leq l-2$
Let

$$\pi : B_l \longrightarrow \mathcal{S}_l$$

be the natural homomorphism where $\mathcal{S}_l$ is the symmetric group on l and define $P_l := \ker\pi$ as the *Pure braid group*.

Let $\mathcal{A}_l = \{z_i = z_j\}_{1 \leq i < j \leq l} = \{H_{ij}\}_{1 \leq i < j \leq l}$ be the arrangement of diagonal hyperplanes in $\mathbb{C}^l$ with complement $M_l = \mathbb{C}^l - \bigcup H_{ij}$, the configuration space of l-ordered points in $\mathbb{C}^l$. $\mathcal{A}_l$ is called **braid arrangement**.

The group P_l can be realized as the fundamental group of the complement M_l of $\mathcal{A}_l$.

$$P_l \cong \pi_1(M_l)$$

$$\mathbf{C} - \{\text{p} - 1 \text{ points}\} \hookrightarrow M_p \longrightarrow M_{p-1}$$

is a linear fibration where $M_p \longrightarrow M_{p-1}$ is the restriction of the map

$$\mathbb{C}^p \longrightarrow \mathbb{C}^{p-1}$$

which forgets the last coordinate.

Properties :

- M_l is a $K(\pi, 1)$-space
- $\pi_1(M_l) \simeq \mathbf{F}_{l-1} \rtimes \mathbf{F}_{l-2} \rtimes \cdots \rtimes \mathbf{F}_2 \rtimes \mathbf{F}_1$ (iterated semi-direct product of free groups)
- $H^*(M_l) \simeq \bigotimes_{k=1}^{l-1} H^1(\bigvee_k S^1)$
- $P_{M_l}(t) = \prod_{k=1}^{l-1} (1 + kt)$

Theorem 1 (Kohno) *Let $\mathcal{A}_l$ be the braid arrangement of $\mathbb{C}^l$; then*

$$\prod_{k \geq 1} (1 - t^k)^{\phi_k(M_l)} = \prod_{1 \leq k \leq l-1} (1 - kt)$$

IV. Arrangements of Hyperplanes

IV.1. Generalities

Let $V = \mathbf{C}^l$ and $\mathcal{A} = \{H_1, \ldots, H_n\}$ be an arrangement of n hyperplanes of V and $M = V - \bigcup_{H \in \mathcal{A}} H$.

$\mathcal{A}$ is said to be *central* if $\bigcap_{H \in \mathcal{A}} H \neq \emptyset$, otherwise it is *affine*.

Let $\mathcal{L}(\mathcal{A})$ be the lattice of the intersections of hyperplanes, ordered by reverse inclusion. Then

$$H^*(M) \simeq A^*(\mathcal{A}) := \bigwedge(e_1, \ldots, e_n)/\mathcal{J}$$

$A^*(\mathcal{A})$ is called the *Orlik-Solomon* algebra where $\mathcal{J}$ is an ideal defined by the dependence relations between hyperplanes of $\mathcal{A}$.

$$P_M(t) = \sum_{X \in \mathcal{L}(\mathcal{A})} \mu(X)(-t)^{codim(X)}$$

which is the Poincaré polynomial of $\mathcal{A}$.

IV.2. Product of arrangements

Let $(\mathcal{A}_1, V_1)$ and $(\mathcal{A}_2, V_2)$ be two arrangements of hyperplanes and $V = V_1 \oplus V_2$. Define the **product** arrangement $(\mathcal{A}_1 \times \mathcal{A}_2)$ by

$$(\mathcal{A}_1 \times \mathcal{A}_2) = \{H_1 \oplus V_2 \mid H_1 \in \mathcal{A}_1\} \cup \{V_1 \oplus H_2 \mid H_2 \in \mathcal{A}_2\}$$

Define
$M_1 := V_1 - \bigcup_{H \in \mathcal{A}_1} H$,
$M_2 := V_2 - \bigcup_{H \in \mathcal{A}_2} H$
and $M := V - \bigcup_{H \in \mathcal{A}_1 \times \mathcal{A}_2} H$ then

$$P_M(t) = P_{M_1}(t) \times P_{M_2}(t)$$

$$\pi_1(M) = \pi_1(M_1) \times \pi_1(M_2)$$

IV.3. Fiber-type arrangements

This is a natural generalization of the braid arrangements.

Definition 2 *$\mathcal{A}$ is a* **fiber-type arrangement** *if there exists a sequence of subarrangements :*

$$\mathcal{A}_1 \subset \mathcal{A}_2 \subset \cdots \subset \mathcal{A}_l = \mathcal{A}$$

such that $|\mathcal{A}_1| = 1$ *and* $M(\mathcal{A}_{i+1}) \longrightarrow M(\mathcal{A}_i)$ *is a locally trivial fibration with fiber* $\mathbf{C} - \{|\,\mathcal{A}_{i+1} - \mathcal{A}_i\,|$ points$\}$.

$\{d_{i+1} := |\,\mathcal{A}_{i+1} - \mathcal{A}_i\,|, 0 \leq i < l-1\}$ where $\mathcal{A}_0 = \emptyset$, is called the set of the *exponents* of $\mathcal{A}$ and $P_M(t) = \prod_i (1 + d_i t)$ for all exponents d_i.

Theorem 3 (Falk, Randell) *The fiber-type arrangements satisfy the LCS formula :*

$$\prod_{k \geq 1} (1 - t^k)^{\phi_k(M)} = \prod_{i=1}^{l} (1 - d_i t)$$

Remarks : Braid arrangements and products of arrangements $\mathcal{A}_1 \times \cdots \times \mathcal{A}_{l-1}$ where $\mathcal{A}_i = \mathbb{C} - \{i$ points$\}$ are fiber-type.
The fundamental groups of the complements of such fiber-type arrangements of hyperplanes cannot be distinguished under lower central series.
Neither homology nor the lower central series can distinguish between Π_l and P_l.

Example 4 *Let be the two arrangements $\mathcal{A}$ and $\mathcal{B}$ defined by :*

$$Q(\mathcal{A}) = xyz(x-y)(x-z)(x-2z)(x-3z)(y-z)(x-y-z)$$

and

$$Q(\mathcal{B}) = xyz(x-y)(x-z)(x-2z)(x-3z)(y-z)(x-y-2z)$$

They are fiber-type with the same exponents $(1,4,4)$. Therefore homology groups and lower central series quotients are isomorphic.

IV.4. Hypersolvable arrangements

The class of hypersolvable arrangements contains fiber-type, *generic* arrangements and many others.

Definition 5 *An arrangement is called* **generic** *iff it is a cone over a general position arrangement.*

Notice that all the fiber-type arrangements are $K(\pi, 1)$ which means that the complement is $K(\pi, 1)$ and the generic arrangements of n hyperplanes in $\mathbb{C}^l$ where $n > l$, are never $K(\pi, 1)$.
There is a "*deformation*" of every hypersolvable arrangement to a fiber-type one with the same intersections up to codimension 2 and therefore with the same fundamental group.

Definition 6 *The* **quadratic** *Orlik-Solomon algebra is defined by*

$$\overline{A}^*(\mathcal{A}) := \bigwedge(e_1, \ldots, e_n)/J$$

where J is the homogeous ideal generated by

$$(\mathcal{R}_\mathcal{A}) \qquad e_{i_1} \wedge e_{i_2} - e_{i_1} \wedge e_{i_3} + e_{i_2} \wedge e_{i_3}$$

with codim$(H_{i_1} \cap H_{i_2} \cap H_{i_3}) = 2$.

Definition 7 *The* **quadratic** *Poincaré polynomial $\overline{P}_M(t)$ is the Poincaré polynomial of $\overline{A}^*(\mathcal{A})$.*

Theorem 8 (Jambu-Papadima) *Let $\mathcal{A}$ be a hypersolvable arrangement. Then*

$$\prod_{k \geq 1} (1 - t^k)^{\phi_k(\mathcal{A})} = \overline{P}_M(-t)$$

(called the generalized LCS formula).

Notice that if $\mathcal{A}$ is fiber-type, then $\overline{A}^*(\mathcal{A}) = A^*(\mathcal{A})$ and $\overline{P}_M(t) = P_M(t)$.

Two remarks :

– **1** Up to now, we have computed the LCS ranks only for hypersolvable arrangements. The formula giving the LCS ranks (LCS formula) uses Poincaré polynomials and as such some numerical invariants of the arrangement.

– **2** The LCS formula does not distinguish fundamental groups of arrangements with the same numerical invariants.

V. Chen Groups

The *Chen groups* of a group G are the lower central series quotients of G modulo its second commutator subgroup G''. Recall that $G^{(i+1)} = [G^{(i)}, G^{(i)}]$, then $G' = [G, G]$ and $G'' = [\Gamma_2 G, \Gamma_2 G]$.
It is finitely-generated if G is finitely-generated.
The k**-th Chen** group of G is, by definition, $\mathrm{gr}_k(G/G'')$.
Let $\theta_k(G) = \phi_k(G/G'')$ be its rank.
The projection $G \longrightarrow G/G''$ induces surjections $\mathrm{gr}_k G \longrightarrow \mathrm{gr}_k(G/G'')$.
Thus $\phi_k \geq \theta_k$ for all k and $\phi_k = \theta_k$ for $k \leq 3$.

Theorem 9 (Chen, Murasugi) *Let* $G = \mathbf{F}_l$. *Then*

$$\theta_k(\mathbf{F}_l) = (k-1).\binom{l+k-2}{k}, \quad k \geq 2$$

Theorem 10 (Cohen, Suciu) *1. Let*

$G = \pi_1(M_1 \times M_2)$, $G_1 = \pi_1(M_1)$, $G_2 = \pi_1(M_2)$

then

$$\theta_k(G) = \theta_k(G_1) + \theta_k(G_2)$$

2. Let $G = \mathbf{F}_{d_1} \times \cdots \times \mathbf{F}_{d_l}$ *be a direct product of free groups, then the Chen groups of* G *are free abelian and*

$$\theta_1(G) = \sum_{i=1}^{l} d_i$$

$$\theta_k(G) = (k-1) \sum_{i=1}^{l} \binom{k + d_i - 2}{k} \text{ for } k \geq 2$$

In particular, let $\Pi_l = \mathbf{F}_{l-1} \times \cdots \times \mathbf{F}_1$ *then*

$$\theta_1(\Pi_l) = \binom{l}{2}$$

$$\theta_k(\Pi_l) = (k-1) \binom{k + l - 2}{k+1} \text{ for } k \geq 2$$

3. The Chen groups of the pure braid group P_l are free abelian. The rank θ_k are given by

$$\theta_1 = \binom{l}{2}, \qquad \theta_2 = \binom{l}{3}$$

and

$$\theta_k = (k-1).\binom{l+1}{4} \text{ for } k \geq 3$$

Corollary 11 *For $l \geq 4$, the groups P_l/P_l'' and Π_l/Π_l'' are not isomorphic. For $l \geq 4$, the groups P_l and Π_l are not isomorphic.*

Remark 12 *$P_2 \simeq \mathbf{F}_1$ and $P_3 \simeq \mathbf{F}_2 \times \mathbf{F}_1$.*

Example 13 *Let be the two arrangements $\mathcal{A}$ and $\mathcal{B}$ defined by :*

$$Q(\mathcal{A}) = xyz(x-y)(x-z)(x-2z)(x-3z)(y-z)(x-y-z)$$

and

$$Q(\mathcal{B}) = xyz(x-y)(x-z)(x-2z)(x-3z)(y-z)(x-y-2z)$$

As we saw, they are fiber-type with the same exponents $(1, 4, 4)$. Therefore homology groups and lower central series quotients are isomorphic.

However,

$$\theta_1(G(\mathcal{A})) = \theta_1(G(\mathcal{B})) = 9$$

$$\theta_2(G(\mathcal{A})) = \theta_2(G(\mathcal{B})) = 12$$

$$\theta_3(G(\mathcal{A})) = \theta_3(G(\mathcal{B})) = 40$$

and for $k \geq 4$

$$\theta_k(G(\mathcal{A})) = \frac{1}{2}(k-1)(k^2+3k+24)$$

$$\theta_k(G(\mathcal{B})) = \frac{1}{2}(k-1)(k^2+3k+22)$$

then

$$G(\mathcal{A}) \not\simeq G(\mathcal{B})$$

Notice that the groups $G(\mathcal{A})$ and $G(\mathcal{B})$ cannot be distinguished by means of the LCS formula.

VI. Resonance Varieties

Let consider the cohomology $H^*(A(\mathcal{A}), d_\omega)$, where d_ω is the degree one mapping defined by left multiplication by a fixed element $\omega = \sum \lambda_i a_i \in A^1(\mathcal{A})$.

The cohomology $H^*(A(\mathcal{A}), d_\omega)$ is isomorphic to the cohomology of the complement $M(\mathcal{A})$ with coefficients in a local system determined by ω, when ω satisfies some *non-resonance* conditions dependent only on $M(\mathcal{A})$.

Then Falk showed that $H^1(A(\mathcal{A}), d_\omega) \neq 0$ precisely when ω belongs to an affine variety called the *resonance variety* $R_1(\mathcal{A})$ of the arrangement $\mathcal{A}$.

$$R_1(\mathcal{A}) = \{\lambda \in \mathbb{C}^n \mid H^1(A(\mathcal{A}), d_\omega) \neq 0\}$$

He showed that $R_1(\mathcal{A})$ is a linear subspace of $\mathbb{C}^n$ which is a union of subspaces of dimension at least 2, as follows.

A partition $\mathsf{P} = (p_1 \mid \cdots \mid p_q)$ of $\mathcal{A}$ is called *neighborly* if

$$p_j \cap I \geq \mid I \mid -1 \Rightarrow I \subset p_j \text{ for all } I \in \mathcal{L}_2(\mathcal{A})$$

To a neighborly partition corresponds an irreducible subvariety of $R_1(\mathcal{A})$

$$L_{\mathsf{P}} = \Delta_n \cap \bigcap_{\{I \in \mathcal{L}_2(\mathcal{A}) \mid I \not\subset p_j, \text{ any } j\}} \{\lambda \mid \sum_{i \in I} \lambda_i = 0\}$$

where $\Delta_n = \{\lambda \in \mathbb{C}^n \mid \sum_{i=0}^{n} \lambda_i = 0\}$.
Conversely, all components of $R_1(\mathcal{A}]$ arise from neighborly partitions of sub-arrangements of $\mathcal{A}$.

This is the first conjecture (*Suciu*) :
Let $R_1(\mathcal{A}) = \bigcup_{i=1}^{v} L_i$ be the decomposition of $R_1(\mathcal{A})$ into linear components.
Then if $\phi_4(\mathcal{A}) = \theta_4(\mathcal{A})$ (which is not true for hypersolvable arrangements)

$$\prod_{k \geq 1} (1 - t^k)^{\phi_k(\mathcal{A})} = \prod_{i=1}^{v} (1 - (\text{dim} L_i) t)$$

VII. Some examples

– **1. Arrangement** $\mathcal{A}_{X_3}$.

Let $Q(\mathcal{A}_{X_3}) = xyz(y+z)(x-z)(2x+y)$ be the defining polynomial of $\mathcal{A}_{X_3}$.

$R_1(\mathcal{A}_{X_3}) = L_{135} \cup L_{236} \cup L_{456}$

and $\dim L_i = 2$ for any L_i.

Moreover $\phi_4(\mathcal{A}_{X_3}) = \theta_4(\mathcal{A}_{X_3}) = 9$, so

$$\prod_{k\geq 1}(1-t^k)^{\phi_k(\mathcal{A}_{X_3})} = \prod_{i=1}^{3}(1-2t) = (1-2t)^3$$

Notice that $\mathcal{A}_{X_3}$ is not hypersolvable.

– **2. Arrangement $\mathcal{A}_{X_2}$.**
Let $Q(\mathcal{A}_{X_2}) = xyz(x+y)(x-z)(y-z)(x+y-2z)$ be the defining polynomial of $\mathcal{A}_{X_2}$.
$R_1(\mathcal{A}_{X_2}) = L_{136} \cup L_{245} \cup L_{127} \cup L_{237} \cup L_{567}$ and $\dim L_i = 2$ for any L_i.
Moreover $\phi_4(\mathcal{A}_{X_2}) = \theta_4(\mathcal{A}_{X_2}) = 15$; so

$$\prod_{k \geq 1} (1-t^k)^{\phi_k(\mathcal{A}_{X_2})} = \prod_{i=1}^{5} (1-2t) = (1-2t)^5$$

Notice that $\mathcal{A}_{X_2}$ is not hypersolvable and it is shown that the LCS formula does not hold.

– **3. Non-Fano arrangement $\mathcal{A}$.**
Let $Q(\mathcal{A}) = xyz(x-y)(x-z)(y-z)(x+y-z)$ be the defining polynomial of $\mathcal{A}$.
$P_M(t) = (1+t)(1+3t)^2$; $\mathcal{A}$ is not hypersolvable (it is called factored) ;
$\phi_4(\mathcal{A}) = 42$ and $\theta_4(\mathcal{A}) = 27$ so the conjecture does not work.

Recall : $R_1(\mathcal{A}) = \bigcup_{i=0}^{v} L_i$ and for any $r \geq 0$.
Let $h_r = |\{L_i \mid \dim L_i = r\}|$ be the number of components of $R_1(\mathcal{A})$ of dimension r.
h_r can be computed directly from the lattice $\mathcal{L}(\mathcal{A})$ by computing neighborly partitions P of sub-arrangements of $\mathcal{A}$ and finding $\dim L_{\mathsf{P}}$.

This is the second conjecture (*Suciu*) :

$$\theta_k(\mathcal{A}) = (k-1) \sum_{r \geq 2} \binom{r+k-2}{k} . h_r$$

for k sufficiently large.